

Universal Lyndon Words

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Abstract. A word w over an alphabet Σ is a Lyndon word if there exists an order defined on Σ for which w is lexicographically smaller than all of its conjugates (other than itself). We introduce and study *universal Lyndon words*, which are words over an n -letter alphabet that have length $n!$ and such that all the conjugates are Lyndon words. We show that universal Lyndon words exist for every n and exhibit combinatorial and structural properties of these words. We then define particular prefix codes, which we call Hamiltonian lex-codes, and show that every Hamiltonian lex-code is in bijection with the set of the shortest unrepeated prefixes of the conjugates of a universal Lyndon word. This allows us to give an algorithm for constructing all the universal Lyndon words.

Keywords: Lyndon word, Universal cycle, Universal Lyndon word, Lex-code.

1 Introduction

A word is called Lyndon if it is lexicographically smaller than all of its conjugate words (other than itself). Lyndon words are an important and well studied object in Combinatorics. Recall, for example, the fact that every Lyndon word is unbordered, or the existence of a unique factorization of any word into a non-decreasing sequence of Lyndon words [5]. The definition of Lyndon word implicitly assumes a lexicographic order. Therefore, for different orders, we typically obtain several distinct Lyndon conjugates of the same word. The motivation of this paper is to push the idea to its limits, and ask whether there is a *universal Lyndon word*, that is, a word of length $n!$ over n letters such that for each of its conjugates there exists an order with respect to which this conjugate is Lyndon.

Such a word resembles similar objects known in the literature as universal cycles. A *universal cycle* [2] is a circular word containing every object of a particular type exactly once as a factor. Probably the most prominent example

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of universal cycles are de Bruijn cycles, which are circular words of length 2^n containing every binary word of length n exactly once.

The set represented by a universal Lyndon word is the set of all total orders on n letters or, equivalently, all permutations of n letters. The most convenient way is to represent the order $a_1 < a_2 < \dots < a_n$ by its “shorthand encoding”, which is the word $a_1 a_2 \dots a_{n-1}$. Jackson [4] showed that the corresponding universal cycles exist for every n and can be obtained from an Eulerian graph in a manner similar to the generation of de Bruijn cycles. Ruskey and Williams [6] gave efficient algorithms for constructing shorthand universal cycles for permutations. Our paper can be seen as a generalization of this concept. Indeed, it is easy to note that every shorthand universal cycle for permutations is a universal Lyndon word (see [3] for more details), but the opposite is not true—that is, there exist universal Lyndon words such that the Lyndon conjugate for some order $a_1 < a_2 < \dots < a_n$ does not start with $a_1 a_2 \dots a_{n-1}$.

We study the structural properties of universal Lyndon words and give combinatorial characterizations. We then develop a method for generating all the universal Lyndon words. This method is based on the notion of Hamiltonian lex-code, which we introduce in this paper.

2 Notation

Given a finite non-empty ordered set Σ (called the *alphabet*), we let Σ^* denote the set of words over the alphabet Σ . Given a finite word $w = a_1 a_2 \dots a_n$, with $n \geq 1$ and $a_i \in \Sigma$, the length n of w is denoted by $|w|$. The *empty word* will be denoted by ε and we set $|\varepsilon| = 0$. We let Σ^n denote the set of words of length n and by Σ^+ the set of non-empty words. For $u, v \in \Sigma^+$ we let $|u|_v$ denote the number of (possibly overlapping) occurrences of v in u . For instance, $|011100|_{00} = 1$ and $|011100|_{11} = 2$.

Given a word $w = a_1 a_2 \dots a_n$, $a_i \in \Sigma$, we say a word $v \in \Sigma^+$ is a *factor* of w if $v = a_i a_{i+1} \dots a_j$ for some integers i, j with $1 \leq i \leq j \leq n$. We let $\text{Fact}(w)$ denote the set of all factors of w and $\text{Alph}(w)$ the set of all factors of w of length 1. If $i = 1$ (resp., $j = n$), we say that the factor v is a *prefix* (resp., a *suffix*) of w . We let $\text{Pref}(w)$ (resp., $\text{Suff}(w)$) denote the set of prefixes (resp., suffixes) of the word w . The empty word ε is a factor, a prefix and a suffix of any word. A factor (resp., a prefix, resp., a suffix) of a word w is *proper* if it is different from ε and from w itself.

A *border* of w is a proper prefix of w that is also a suffix of w . A word is said to be *unbordered* if it does not have borders. A word u is a *cyclic factor* of w if $u \in \text{Fact}(ww)$ and $|u| \leq |w|$. We let $|w|_u^c$ denote the number of (possibly overlapping) occurrences of u as a cyclic factor of w . For instance, $|011100|_{00}^c = 2$. We say that a word u is *conjugate* to a word v if there exist words w_1, w_2 such that $u = w_1 w_2$ and $v = w_2 w_1$. The conjugate is *proper* if both w_1 and w_2 are non-empty. The conjugacy being an equivalence relation, we can define a *cyclic word* as a conjugacy class of words. Note that u is a cyclic factor of a word w if and only if u is a factor of a conjugate of w .

Every total order on the alphabet Σ induces a different lexicographic (or dictionary) order on Σ^* . Recall that the lexicographic order \triangleleft on Σ^* induced by the order $<$ on the alphabet Σ is defined as follows: $u \triangleleft v$ if u is a prefix of v or za is a prefix of u and zb is a prefix of v , with $a < b$. We say that a word w over Σ is a *Lyndon word* if there exists a total order on Σ such that, with respect to this order w is lexicographically smaller than all of its proper conjugates (or, equivalently, proper suffixes). For example, the word $w = abcabb$ is a Lyndon word, because for the order $a < c < b$ it is the smallest word in its conjugacy class. Note that a Lyndon word must be primitive (i.e., it cannot be written as a concatenation of two or more copies of a shorter word), and therefore its conjugates are all distinct.

A set of words $X \subset \Sigma^+$ is a *code* if for every $x_1, x_2, \dots, x_h, x'_1, x'_2, \dots, x'_k \in X$, if $x_1 x_2 \dots x_h = x'_1 x'_2 \dots x'_k$, then $h = k$ and $x_i = x'_i$ for every $1 \leq i \leq h$. For example, $X = \{ab, abb\}$ is a code. Every set $X \subset \Sigma^+$ with the property that no word in X is a prefix of another word in X is a code, and is called a *prefix code*.

A *directed graph* (or *digraph*) is a pair $G = (V, E)$, where V is a set, whose elements are called *vertices*, and E is a binary relation on V (i.e., a set of ordered pairs of elements of V) whose elements are called *edges*. The *indegree* (resp., *outdegree*) of a vertex v in a digraph G is the number of edges incoming to v (resp., outgoing from v). A *walk* in a digraph G is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \dots e_{k-1} v_k$ of vertices and edges of G such that $e_i = (v_i, v_{i+1})$ for every $i < k$. If $v_0 = v_k$ the walk is *closed*. A closed walk in a digraph G is an *Eulerian cycle* if it traverses every edge of G exactly once. A digraph is *Eulerian* if it admits an Eulerian cycle. A fundamental property of graphs is that a connected digraph is Eulerian if and only if the indegree of each vertex is equal to its outdegree. A closed walk in a digraph G is a *Hamiltonian cycle* if it contains every vertex of G exactly once. A digraph is *Hamiltonian* if it admits a Hamiltonian cycle.

In the rest of the paper, we let Σ_n denote the alphabet $\{1, 2, \dots, n\}$, $n > 0$.

3 Universal Lyndon Words

Definition 1. A *universal Lyndon word (ULW)* of degree n is a word over Σ_n that has length $n!$ and such that all its conjugates are Lyndon words.

Remark 1. Since there exist $n!$ possible orders on Σ_n , a universal Lyndon word w of degree n has the property that for every order on Σ_n , there is exactly one conjugate of w that is Lyndon with respect to this order; on the other hand, from the definition it follows that a conjugate of a universal Lyndon word cannot be Lyndon for more than one order.

We consider universal Lyndon words up to rotation, i.e., as cyclic words.

Example 1. The only universal Lyndon word of degree 1 is 1, and the only universal Lyndon word of degree 2 is 12. There are three universal Lyndon words of degree 3, namely 212313, 323121 and 131232. Note that these words

are pairwise isomorphic (i.e., one can be obtained from another by renaming letters). There are 492 universal Lyndon words of degree 4. There are 41 if we consider them up to isomorphism, and are presented in Tables 1 and 2.

Remark 2. It is worth noticing that a universal Lyndon word cannot contain a square (i.e., a concatenation of two copies of the same word) as a cyclic factor. That is, a universal Lyndon word is cyclically square-free. Indeed, if uu is a factor of w , then there is a conjugate of w that has u as a border, and it is easily shown that every Lyndon word must be unbordered, and therefore no conjugate of a universal Lyndon word can have a border.

The following proposition gives a sufficient condition for a word being a ULW.

Proposition 1. *Let $n \geq 2$, and w be a word over Σ_n such that every permutation of $n - 1$ elements of Σ_n appears as a cyclic factor in w exactly once. Then w is a universal Lyndon word.*

Proof. Suppose that every permutation of $n - 1$ elements of Σ_n appears as a cyclic factor in w exactly once. Since there are $n!$ such words, this implies that w has length $n!$. Now, for any order $a_1 < a_2 < \dots < a_{n-1} < a_n$ over Σ_n , there is exactly one conjugate of w beginning with $a_1 a_2 \dots a_{n-1}$, and this conjugate is Lyndon with respect to this order. So w has exactly $n!$ distinct Lyndon conjugates and therefore is a universal Lyndon word. \square

Remark 3. One might wonder whether it is sufficient to suppose that *each* of w 's factors of length $n - 1$ appears exactly once in the word w to guarantee that w is a ULW. This is not the case. For example, let $n = 4$; the word $w = 123412431324134214231432$ has $n!$ distinct factors of length $n - 1$ but is not a universal Lyndon word, since its conjugate 314321234124313241342142 is not Lyndon for any order (in fact this is a consequence of the fact that the conjugate 313241342142314321234124 is Lyndon both for the orders $3 < 1 < 2 < 4$ and $3 < 1 < 4 < 2$).

We now use the result of Proposition 1 to show that there exist universal Lyndon words for each degree.

Given an integer $n > 2$, the *Jackson graph of degree n* , denoted $J(n)$, is a directed graph in which the nodes are the words over Σ_n that are permutations of $n - 2$ letters, and there is an edge from node u to node v if and only if the suffix of length $n - 3$ of u is equal to the prefix of length $n - 3$ of v and the first letter of u is different from the last letter of v . The label of such an edge is set to the first letter of u . In Fig. 1, the Jackson graph $J(4)$ is depicted.

Proposition 2. *There exist universal Lyndon words of degree n for every $n > 0$.*

Proof. We can suppose $n > 2$. Take the Jackson graph $J(n)$. By construction, this graph is connected and the indegree and outdegree of each vertex are both equal to 2. Therefore, it contains an Eulerian cycle. Let w denote the word obtained by concatenating the labels of such an Eulerian cycle. Note that every

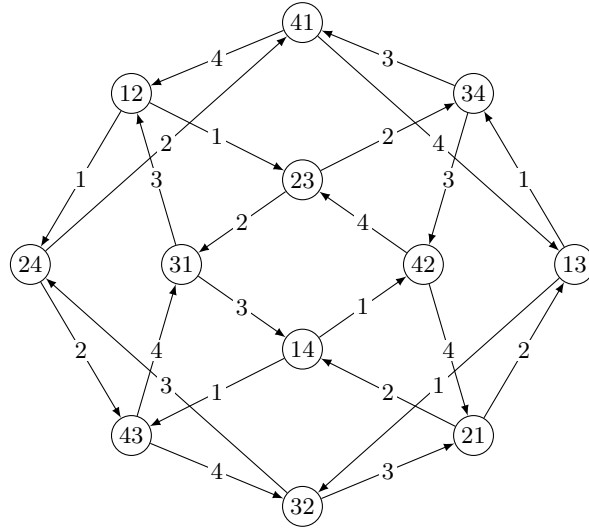


Fig. 1. The Jackson graph $J(4)$ of degree 4. Every Eulerian cycle of $J(4)$ is a universal Lyndon word.

word that is the permutation of $n - 2$ letters appears as a cyclic factor in w exactly twice and the two occurrences are followed by the two letters that do not appear in the factor. By Proposition 1, w is then a universal Lyndon word of degree n . \square

A universal Lyndon word that is an Eulerian cycle of a Jackson graph is called a *universal cycle* [4], or *shorthand universal cycle for permutations* [6], but in this paper we will call it a *universal Lyndon word of Jackson type*, or simply a *Jackson universal Lyndon word*.

The Jackson universal Lyndon words of degree 4 are presented in Table 1 (the list contains only pairwise non-isomorphic words, in their representation starting with 1231).

However, there are universal Lyndon words that are not of Jackson type. In fact, the converse of Proposition 1 is not true. For instance, $w = 123431242314132421343214$ is a universal Lyndon word of degree 4 but it does not contain any of 142, 143, 241, 243, 341, 342 as a factor.

4 Order-defining Words

In this section, we give combinatorial results on the structure of universal Lyndon words.

123124132431432142342134	123124132134214324314234
123124314214321324134234	123124134213243214314234
123124132431421432134234	123124314234213214324134
123124314213214324134234	123124321431423421324134
123124132431423421432134	123124134213243143214234
123124314214324132134234	123124132431432134214234
123124132432143142342134	123124321342143142341324
123124132431432142134234	123124132143243142134234
123124314234132134214324	123124132432143142134234
123124314234134214321324	123124134214321324314234

Table 1. The 20 Jackson universal Lyndon words of degree 4, up to isomorphisms.

Let $w = a_1 a_2 \cdots a_n!$ be a universal Lyndon word of degree n . Let w_i denote the conjugate of w starting at position i , that is,

$$w_i = a_i a_{i+1} \cdots a_n! a_1 a_2 \cdots a_{i-1}.$$

Definition 2. We say that a partial order \triangleleft on Σ_n is a partial alphabet order with respect to $I \subseteq \Sigma_n$ if \triangleleft is a total order on I , $i \triangleleft j$ for each $i \in I$ and $j \in \Sigma_n \setminus I$, and all $j, k \in \Sigma_n \setminus I$ are incomparable. The size of \triangleleft is set to $|I|$.

Note that a partial alphabet order of size $n - 1$ is a total order on Σ_n .

Every word $u \in \Sigma_n^+$ defines a partial alphabet order \triangleleft_u with respect to $\text{Alph}(u)$, defined as follows: $i \triangleleft_u j$ if and only if the first occurrence of i in u precedes the first occurrence of j in u .

The following proposition shows that in a universal Lyndon word, every conjugate is Lyndon with respect to the order it defines. This is an important structural property of universal Lyndon words, which is not true in general. Take, for example, the word $w = 123122$. It is Lyndon with respect to the order $1 < 3 < 2$, but it is not Lyndon with respect to the order it defines, $1 < 2 < 3$.

We let \triangleleft_i denote the order defined by w_i and by \blacktriangleleft_i the order with respect to which w_i is Lyndon.

Theorem 1. Let w be a word of length $n!$ over Σ_n . Then w is a ULW if and only if every conjugate of w is Lyndon with respect to the order it defines. That is, $\triangleleft_i = \blacktriangleleft_i$ for every i .

Proof. If every conjugate of w is Lyndon, then w is ULW by definition. So we only have to prove the “only if” part of the statement.

Suppose that $\triangleleft_j \neq \blacktriangleleft_j$ for some j , and let k be such that $\blacktriangleleft_k = \triangleleft_j$. Let z be the longest common prefix of w_j and w_k . Then za is a prefix of w_j and zb a prefix

of w_k , where $a \neq b$ are letters. We have $a \triangleleft_j b$ and $b \triangleleft_k a$. Therefore, also $b \triangleleft_j a$, which implies that there exists $u \in \Sigma_n^*$ such that bua is a suffix of za and bub is a suffix of zb . Let w_ℓ be the conjugate starting with bua . Obviously, $b \triangleleft_\ell a$, since b is the first letter of w_ℓ . But then we have that $bub \triangleleft_\ell bua$, and therefore w_ℓ has a conjugate smaller than itself for the order \triangleleft_ℓ , a contradiction. \square

Proposition 3. *Let w be a universal Lyndon word, and u a cyclic factor of w . Then for every conjugate w_i of w , we have that u is a prefix of w_i if and only if $\triangleleft_u \subseteq \triangleleft_i$.*

Proof. By Theorem 1, we have $\triangleleft_u \subseteq \triangleleft_i$ for each i such that u is a prefix of w_i . Choose one such w_i (which exists since u is a cyclic factor of w). Let $\triangleleft_u \subseteq \triangleleft_j$ and suppose that za is a prefix of u and zb a prefix of w_j for two distinct letters a and b and some $z \in \Sigma_n^*$. Then $a \triangleleft_i b$, and, since $a \in \text{Alph}(u)$, we deduce that $a \triangleleft_u b$. This implies that $a \triangleleft_j b$, since $\triangleleft_u \subseteq \triangleleft_j$. Therefore, $w_i \triangleleft_j w_j$, a contradiction. \square

Proposition 3 states that the cyclic factors of a ULW are in one-to-one correspondence with the orders they define. As an example, if $1 < 2$ and, say, 212 is a cyclic factor of a universal Lyndon word w , then every other occurrence of 21 in w must be followed by 2.

Corollary 1. *Let w be a universal Lyndon word of degree n , and u a cyclic factor of w of length $k > 0$. Then u is the lexicographically smallest cyclic factor of w of length k with respect to any total order \triangleleft on Σ_n such that $\triangleleft_u \subseteq \triangleleft$.*

We now give a combinatorial characterization of universal Lyndon words.

Theorem 2. *Let w be a word over Σ_n . Then w is a universal Lyndon word if and only if for every cyclic factor u of w , one has*

$$|w|_u^c = (n - |\text{Alph}(u)|)! \quad (1)$$

Proof. Suppose that w is a ULW. There are $(n - |\text{Alph}(u)|)!$ many total orders \triangleleft on Σ_n such that $\triangleleft_u \subseteq \triangleleft$. Hence, (1) follows from Corollary 1.

Suppose now that (1) holds for every cyclic factor u of w and let us prove that w is a ULW. For every letter $a \in \Sigma_n$, one has $|w|_a = |w|_a^c = (n - 1)!$, so that $|w| = \sum_{a \in \Sigma_n} |w|_a = n!$. Moreover, w is primitive, since $|w|_w^c = 1$. We show that w is a Lyndon word with respect to \triangleleft_w . Let v be a proper conjugate of w and let z be the longest common prefix of w and v . Let a and b be the letters that follow the prefix z in w and v respectively. Since both za and zb occur in w , we have $|w|_z^c > |w|_{zb}^c$ which implies $b \notin \text{Alph}(z)$ by (1). Because za is a prefix of w and $b \notin \text{Alph}(z)$, one has $a \triangleleft_w b$, and therefore $w \triangleleft_w v$. This proves that w is a Lyndon word. By a similar argument, all conjugates of w are Lyndon words, so that w is a ULW. \square

Corollary 2. *The reversal of a ULW is a ULW.*

Note that the fact that the set of universal Lyndon words is closed under reversal is not an immediate consequence of the definition. This property is not true for Lyndon words, e.g. the word 112212 is Lyndon but its reversal is not.

Definition 3. We say that u is a minimal order-defining word if no proper factor of u defines \triangleleft_u .

Proposition 4. Given a universal Lyndon word w of degree n , for each partial alphabet order \triangleleft on Σ_n there is a unique minimal order-defining word with respect to \triangleleft that is a cyclic factor of w .

Proof. Let \triangleleft be a partial alphabet order with respect to I . Let w_i be such that $\triangleleft \subseteq \triangleleft_i$, and let u be the shortest prefix of w_i such that $\text{Alph}(u) = I$. Note that $\triangleleft_u = \triangleleft$ by Theorem 1. Clearly, u is a minimal order-defining word, and the uniqueness is a consequence of Proposition 3. \square

Let w be a universal Lyndon word. We let $MT(w)$ denote the minimal total order-defining words of w , i.e., the set of cyclic factors of w that are minimal order-defining words with respect to a total order on Σ_n . The next proposition is a direct consequence of the definitions and of the previous results.

Proposition 5. Let w be a universal Lyndon word of degree n , and u a cyclic factor of w . The following conditions are equivalent:

1. $u \in MT(w)$;
2. $|\text{Alph}(u)| = n - 1$, and $|\text{Alph}(u')| < n - 1$ for each proper prefix u' of u ;
3. there exists a unique conjugate w_i of w such that u is the shortest unrepeated prefix of w_i .

The shortest unrepeated prefix of a word is also called its *initial box* [1].

In what follows, we exhibit a structural property of ULW.

Definition 4. We say that a cyclic factor v of a word w is a stretch if w contains a cyclic factor avb with $a, b \in \Sigma_n \setminus \text{Alph}(v)$. Let u be a cyclic factor of w . We say that a cyclic factor v of w is a stretch extension of u in w if u is a factor of v , $\text{Alph}(u) = \text{Alph}(v)$, and v is a stretch.

Of course, a stretch is always a stretch extension of itself.

Example 2. Let $w = 123412431324134214231432$. Then 31 has two stretch extensions in w , namely 313 and itself.

Lemma 1. Each cyclic factor u of a ULW w has a unique stretch extension in w . Moreover, it has a unique occurrence in its stretch extension.

Proof. Let v be a stretch extension of u in w . Then u and v have the same number of cyclic occurrences in w by Theorem 2. \square

Theorem 3. If asa is a cyclic factor of a ULW w , with $a \in \Sigma_n \setminus \text{Alph}(s)$, then bsb is a cyclic factor of w for each $b \in \Sigma_n \setminus \text{Alph}(s)$.

Proof. Proceed by induction on $|s|$. The claim trivially holds for $|s| = 0$, since aa is not a cyclic factor of w . Let now $|s| > 0$. We first show that if bs is a cyclic factor of w , then also bsb is a cyclic factor of w . Let therefore bs be a cyclic factor of w , where $b \neq a$ is a letter, and let j be such that $\triangleleft_{bsa} \subseteq \blacktriangleleft_j$. By Lemma 1, the word bsa is not a prefix of w_j . Let therefore $bs'e$ be a prefix of w_j and $bs'f$ a prefix of bsa where e and f are distinct letters. Suppose first that $e = b$. If $s' = s$, then bsb is a cyclic factor of w as required. If, on the other hand, the word s' is a proper prefix of s , then the induction assumption for the word $bs'b$ implies that $as'a$ is a cyclic factor of w . This is a contradiction with Proposition 3 since $\triangleleft_{as'a} \subseteq \triangleleft_{asa}$. Let now $e \neq b$. Note that then $\triangleleft_{s'e} \subseteq \triangleleft_{sa}$ since $\triangleleft_{bsa} \subseteq \blacktriangleleft_j$. But we have also $\triangleleft_{s'f} \subseteq \triangleleft_{sa}$, a contradiction with Proposition 3.

The proof is concluded by a counting argument. Theorem 2 implies that, for any $b \notin \text{Alph}(s)$, the word s has m times more cyclic occurrences in w than bsb , where m is the cardinality of $\Sigma_n \setminus \text{Alph}(s)$. \square

The previous result shows the combinatorial structure of universal Lyndon words. Note that the factors of the form asa , $a \in \Sigma_n \setminus \text{Alph}(s)$, with $|\text{Alph}(s)| < n - 2$, only appear in non-Jackson universal Lyndon words. In fact, they can be viewed as premature repetitions of the letter a .

5 Universal Lyndon Words and Lex-codes

Proposition 1 implies that an Eulerian cycle in a Jackson graph is a universal Lyndon word. However, there exist universal Lyndon words that do not arise from a Jackson graph, as we showed at the end of Section 3.

The non-Jackson universal Lyndon words of degree 4 are presented in Table 2 (the list contains only pairwise non-isomorphic words, in their representation starting with 2123).

We now exhibit a method for constructing all the universal Lyndon words. This method is based on particular prefix codes, whose definition is given below.

Definition 5. A set $X \subseteq \Sigma_n^*$ is a *lex-code of degree n* if:

1. for any $x \in X$, there exists a unique ordering of Σ_n such that x is the lexicographical minimum of X ;
2. if u is a proper prefix of some word of X , then u is a prefix of at least two distinct words of X .

A lex-code X of degree n is *Hamiltonian* if the relation

$$S_X = \{(x, y) \in X \times X \mid \exists a \in \Sigma, x \text{ is a prefix of } ay\}$$

has a Hamiltonian digraph.

Notice that Condition 1 in the previous definition ensures that a lex-code is a prefix code.

The following theorem shows the relationships between Hamiltonian lex-codes and universal Lyndon words.

212313243134212414234143	212313241432124313414234
212313241423414321243134	212313241432124313423414
212313421243132414234143	212313414234212431324143
212313241421243134234143	212341423132414321243134
212313241423414313421243	212313212414324313414234
212313243134142342124143	212313212414324313423414
212313212432414234143134	212313414234212414313243
212313212414234143243134	212313212432414313423414
212313212431342341432414	212313243212414313423414
212313241431342341421243	212313212432414313414234
212313212431341432414234	

Table 2. The 21 non-Jackson universal Lyndon words of degree 4, up to isomorphisms.

Theorem 4. *Let w be a ULW. Then the set $MT(w)$ is a Hamiltonian lex-code. Conversely, if $X \subseteq \Sigma_n^*$ is a Hamiltonian lex-code, then there exists a ULW w such that $X = MT(w)$.*

Proof. We assume that w is a ULW and show that $MT(w)$ verifies the definition of lex-code. Since there is a bijection between the elements of $MT(w)$, the conjugates of w (Proposition 5) and the total orders on Σ_n (Theorem 1), Condition 1 is a direct consequence of Corollary 1. Always from Proposition 5, any proper prefix x' of a word x in $MT(w)$ contains less than $n - 1$ distinct letters. From Theorem 2, x' has at least two occurrences as a cyclic factor of w . Therefore, there exist at least two distinct conjugates w_i and w_j of w beginning with x' . Then x' is a proper prefix of the shortest unrepeated prefixes of w_i and w_j respectively. By Proposition 5, we conclude that Condition 2 holds.

Now, we show that the lex-code X is Hamiltonian. For every $0 \leq i \leq n! - 1$, let a_i be the first letter of the conjugate w_i of w . Notice that for every $0 \leq i \leq n! - 2$ one has $a_i w_{i+1} = w_i a_i$. By Proposition 5, every word in $MT(w)$ is the shortest unrepeated prefix x_i of a conjugate w_i . As x_{i+1} is an unrepeated prefix of w_{i+1} , the word $v = a_i x_{i+1}$ is an unrepeated prefix of $a_i w_{i+1} = w_i a_i$. Thus, either $v = w_i a_i$ or v is an unrepeated prefix of w_i . In both cases, x_i is a prefix of v and therefore $(x_i, x_{i+1}) \in S_X$. Similarly, one has $(x_{n!-1}, x_0) \in S_X$. We conclude that $(x_0, x_1, \dots, x_{n!-1}, x_0)$ is a Hamiltonian cycle in the digraph of S_X .

Conversely, we assume that X is a Hamiltonian lex-code and show that $X = MT(w)$ for a suitable ULW w . Let $(x_0, x_1, \dots, x_{k-1}, x_0)$ be a Hamiltonian cycle in the digraph of S_X . By Condition 1, one has $k = n!$ and X is a prefix code. Since $(x_i, x_{i+1}) \in S_X$, $0 \leq i < k$ (where $x_k = x_0$) one has

$$x_i u_i = a_i x_{i+1} \quad (2)$$

for suitable $a_i \in \Sigma_n$, $u_i \in \Sigma_n^*$, $0 \leq i < k$.

Set $w_i = a_i \cdots a_{k-1} a_0 \cdots a_{i-1}$, $0 \leq i < k$. By iterated application of (2), one obtains that x_i is a prefix of a power of w_i . Now let $0 \leq i, j < k$ and $i \neq j$. For a sufficiently large m , x_i, x_j are prefixes of w_i^m, w_j^m , respectively. Thus, taking into account that X is a prefix code, for every total order \triangleleft on Σ_n , one has $w_i \triangleleft w_j$ if and only if $x_i \triangleleft x_j$. From this remark, in view of Condition 1, one derives that $w = w_0$ is a ULW.

To complete the proof, it is sufficient to show that x_i is the shortest unrepeated prefix of w_i , $0 \leq i < k$. In fact, this implies that $X = MT(w)$. Suppose that the shortest unrepeated prefix h_i of w_i is a proper prefix of x_i . Then by Condition 2, h_i is also prefix of x_j and consequently of w_j , for some $j \neq i$. But this contradicts Proposition 5. Thus x_i is a prefix of h_i . Now, suppose $x_i \neq h_i$. Since by Proposition 5, h_i is a shortest word containing $n - 1$ distinct letters, $|\text{Alph}(x_i)| < n - 1$ and, by Theorem 2, x_i has at least another occurrence starting at a position $j \neq i$. So we have that the words x_i and x_j are one a prefix of the other, against the fact that X is a prefix code. \square

From Theorem 4, in order to produce a ULW, one can construct a lex-code and check whether it is Hamiltonian. Let S_n be the set of the total orders on Σ_n . All lex-codes of degree n can be obtained by a construction based on iterated refinements of a partition of S_n as follows:

1. set $X = \{\varepsilon\}$ and $C_\varepsilon = S_n$;
2. repeat the following steps until C_x is a singleton for all $x \in X$:
 - (a) select $x \in X$ such that C_x contains at least two elements;
 - (b) choose $\Gamma \subseteq \Sigma_n$;
 - (c) for any $a \in \Gamma$, let C_{xa} be the set of the orders of C_x such that $a = \min \Gamma$;
 - (d) replace X by $(X \setminus \{x\}) \cup \{xa \mid a \in \Gamma, C_{xa} \neq \emptyset\}$.

An example of execution of the previous algorithm is presented in Ex. 3.

One can verify that after each iteration of loop 2, X is a prefix code, $(C_x)_{x \in X}$ is a partition of S_n , and any $x \in X$ is the lexicographic minimum of X for all orders of C_x . It follows that the procedure halts when X is a lex-code. Moreover, one can prove that any lex-code X may be obtained by the procedure above, choosing conveniently Γ at step (b) of each iteration.

Clearly, not all lex-codes are Hamiltonian. Thus, the main problem is to understand which limitations the Hamiltonianicity of the lex-code imposes to the construction above. For example, the words in a lex-code can be arbitrarily long. But by Theorem 4, if X is a lex-code of degree n and $u \in X$ is longer than $n!$, then X cannot be Hamiltonian.

Example 3. Let $n = 3$. At the beginning of the algorithm, $X = \{\varepsilon\}$ and $C_\varepsilon = S_3 = \{123, 132, 213, 231, 312, 321\}$. The first choice of a word x in X is forced, we must take $x = \varepsilon$. Let us choose $\Gamma = \{1, 2\}$. We then get $C_1 = \{123, 132, 312\}$, $C_2 = \{213, 231, 321\}$ and X becomes $\{1, 2\}$. Let us now choose $x = 1$ and $\Gamma = \{1, 3\}$. We get $C_{11} = \{123, 132\}$, $C_{13} = \{312\}$ and therefore $X = \{2, 11, 13\}$. Next, take $x = 2$ and $\Gamma = \{2, 3\}$; now $C_{22} = \{213, 231\}$, $C_{23} = \{321\}$ and

$X = \{11, 13, 22, 23\}$. Then pick $x = 11$ and $\Gamma = \{2, 3\}$, so that $C_{112} = \{123\}$, $C_{113} = \{132\}$ and $X = \{13, 22, 23, 112, 113\}$. Finally, the last choice of a word in X is forced, $x = 22$ (since C_{22} is the only set of cardinality greater than 1 left). We choose $\Gamma = \{1, 3\}$ and get $C_{221} = \{213\}$ and $C_{223} = \{231\}$. The lex-code obtained is thus $X = \{13, 23, 112, 113, 221, 223\}$. The reader can verify that this lex-code is not Hamiltonian.

The following choices of x and Γ lead to the Hamiltonian lex-code $X = \{12, 13, 21, 23, 31, 32\}$: $x = \varepsilon$, $\Gamma = \{1, 2, 3\}$; $x = 1$, $\Gamma = \{2, 3\}$; $x = 2$, $\Gamma = \{1, 3\}$; $x = 3$, $\Gamma = \{1, 2\}$.

6 Conclusion and Open Problems

We introduced universal Lyndon words, which are words over an n -letter alphabet having $n!$ Lyndon conjugates. We showed that this class of words properly contains the class of shorthand universal cycles for permutations. We gave combinatorial characterizations and constructions for universal Lyndon words. We leave open the problem of finding an explicit formula for the number of ULW of a given degree.

We exhibited an algorithm for constructing all the universal Lyndon words of a given degree. The algorithm is based on the search for a Hamiltonian cycle in a digraph defined by a particular code, called Hamiltonian lex-code, that we introduced in this paper. It would be natural to find efficient algorithms for generating (or even only counting) universal Lyndon words.

Finally, universal Lyndon words have the property that every conjugate defines a different order, with respect to which it is Lyndon. We can define a *universal order word* as a word of length $n!$ over Σ_n such that every conjugate defines a different order. Universal Lyndon words are therefore universal order words, but the converse is not true, e.g. the word 123421323121424314324134 is a universal order word but is not ULW. Thus, it would be interesting to investigate which properties of universal Lyndon words still hold for this more general class.

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